

STUDENTS' GENERALIZATIONS IN THE DEVELOPMENT OF NON-LINEAR MEANINGS OF MULTIPLICATION AND NON-LINEAR GROWTH

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The study reported on in this paper is an interview study conducted with 20 7th and 8th grade students whose purpose was to understand the generalizations they could make about non-linear meanings of multiplication (NLMM) and non-linear growth (NLG) in the context of solving combinatorics problems. The paper identifies productive challenges for the students, and thus fruitful areas where the students could generalize their reasoning about NLMM and NLG.

Keywords: Cognition; Number Concepts and Operations; Middle School Education; Algebra and Algebraic Thinking

As students progress into the middle grades they are expected to begin to understand situations that involve NLMM and NLG. Despite this curricular structure, studies in this area have highlighted the difficulties that students have in reasoning about a broad range of contexts that can involve NLMM and NLG (Van Dooren, De Bock, Janssens & Verschaffel, 2008). These difficulties include a tendency for students to generalize linear meanings of multiplication (LMM) and linear growth (LG) to situations that involve NLMM and NLG. For example, students frequently conclude that scaling the side lengths of a square by a factor of k produces a change in the area of the square by a factor of k rather than k^2 (DeBock, Van Dooren, Janssens, & Verschaffel, 2007; Vlahovic-Stetic, Pavlin-Bernardic, & Rajter, 2010), and treat non-linear functions, for example quadratic functions, as if they have similar properties as linear functions (Chazan, 2006; Ellis & Grinstead, 2008; Zaslavsky, 1999). Thus, an important issue for researchers to investigate is how to support middle grades students to begin to establish NLMM and NLG, and further, to identify productive generalizations students can make in the context of establishing NLMM and NLG (Ellis, 2011). The purpose of this study, which is currently underway, was to address these two issues.

We addressed these issues by presenting 20 7th and 8th grade students with combinatorics problems whose solution could be represented with a two-dimensional array. Thus students had the potential to establish NLMM and NLG, and generalizations about NLMM and NLG as a result of reasoning about relationships among one and two-dimensional quantities. Here we report results from the first of two interviews. The purpose of the first interview was twofold: (a) to establish which of three qualitatively distinct multiplicative concepts students were using (Author, 2009; Steffe, 1994); and (b) to identify productive challenges in the domain of NLMM and NLG for students that were using each multiplicative concept. Once productive challenges were identified for students using each of the three multiplicative concepts this information was used to design tasks for the second interview whose purpose was to examine the generalizations that students using the three different multiplicative concepts could make.

Perspectives and Theoretical Framework

A Quantitative Approach to Non-Linear Meanings of Multiplication and Growth

We developed NLMM and NLG using combinatorics problems like the *Digit Problem*.

Digit Problem: You have a deck of cards with the digits one through seven written in black on them. You draw a card, replace it, and draw a second card. How many possible coordinate points (e.g., (1,7) is one coordinate point) could you make using this process?

We specifically selected combinatorics problems to investigate these issues because of their potential to support students to establish one and two-dimensional quantities, and relationships between them.

That is, combinatorics problems can involve *ordering* the digits in the deck of cards (a first digit, second digit, etc.), and *ordering* the draws from the deck of cards (a first draw, and a second draw) (Author, 2013). We considered this to be a potential basis for spatially structuring the units on the axes, and the axes themselves, which in turn could create a spatial structuring for the coordinate points in the array. Further, we considered that these problems could involve *pairing* a digit from the first draw with a digit from the second draw (Author, 2013), and thus give students an operative way to create what we considered two dimensional units (i.e., coordinate pairs) from one dimensional units (i.e., digits).

We considered that the solution of the digit problem could involve students in establishing a meaning for 7^2 , an NLMM (Figure 1a). Once students solved this initial problem we were interested in introducing NLG by having students consider how changing the number of digits would change the number of coordinate points, which we considered to involve whole number co-variation¹. We wanted to investigate this issue through two types of problems: (a) having students consider how adding additional digits to the deck of cards (e.g., adding the digit 8, then adding the digit 9) would change the total number of coordinate points by considering how many *new* coordinate points there would be after a given change in digits; and (b) having students consider how doubling, tripling, etc. the number of digits would change the number of coordinate points. The goal of the first task was for students to have an experience of NLG by investigating how *equal* changes in the number of digits (e.g., changing the number of digits from 7 to 8, and then from 8 to 9) yielded growth in the number of coordinate points that was *not constant* (e.g., 15 new coordinate points when the number of digits increased from 7 to 8, and 17 new coordinate points when the number of digits increased from 8 to 9) (Figure 1b).

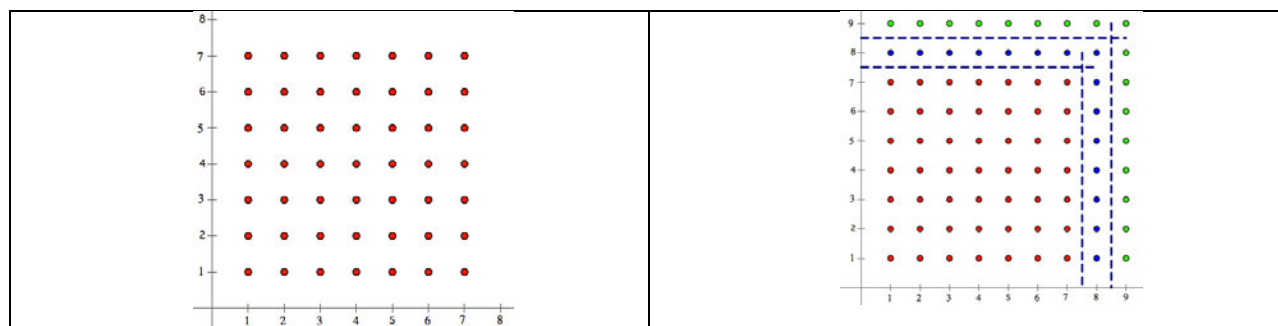


Figure 1a (left) & 1b (right): Arrays for the Digit Problem

The goal of the second task was for students to have an experience of NLG by investigating how multiplicatively increasing the number of digits (e.g., doubling) increased the number of coordinate points by the square of the increase in the number of digits (e.g., quadrupled) (Figure 2). Both of these problems we considered important to students' initial understandings of NLG.

Discussion of Students' Multiplicative Concepts

Prior research has identified three qualitatively distinct multiplicative concepts, all of which are rooted in students' units-coordinating activity (Author, 2009; Steffe, 1992, 1994). We outline the second and third of these multiplicative concepts because we report on data related to students using each concept. We use the Candy Problem to outline the concepts:

Candy Problem: Brandy has 3 packages of candy each containing 6 candies. How many candies does she have in all?

A student using the second multiplicative concept (MC2) has interiorized two levels of units, which enables her to strategically reason with sixes in solving the Candy Problem. For example, to solve the candy problem a student might reason that six and four is ten, and that two more is twelve

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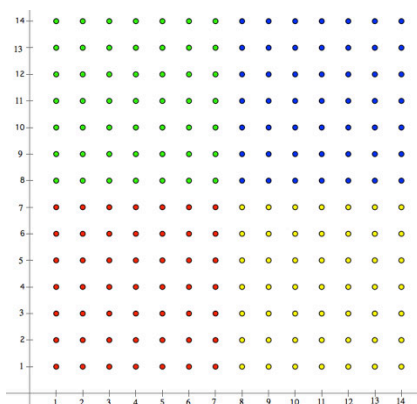


Figure 2: Array for the Digit Problem

and then finish the solution by reasoning that six more than twelve is eighteen. The ability to operate on each six by breaking it into, for example, four and two and strategically combining the parts is a hallmark of MC2 students because it is one indication that they are able to operate on a unit of six units. MC2 students are also able to produce three levels of units in activity. That is, once they have solved the Candy Problem they may regard the 18 candies as a unit of three units of six units, but they cannot use this three level of unit structure in further operating. For example, if these students were told that Brandy received 8 more packages, they could determine that this was 48 more candies, producing 48 as a unit of 8 units of 6 units. They could then unite 18 candies and 48 candies to determine there were 66 candies in all. However, determining that 66 candies was 11 packages would be a separate problem for an MC2 students because they would not retain the 18 candies as a unit of 3 units of 6 units and the 48 candies as a unit of 8 units of 6 units. This means that MC2 students are able to create three levels of units in activity, but are not able to operate on this third level of unit.

In contrast, students using the third multiplicative concept (MC3) have *interiorized* three levels of units. This means that an MC3 student would be able to determine that the 66 candies constituted 11 packages because the 18 candies would retain there status as a unit of 3 units of 6 units, and the 48 candies would retain there status as a unit of 8 units of 6 units. So when they united 18 and 48 they would establish the 66 candies as a unit of 11 units of 6 units.

Methods

The data for this research was drawn from clinical interviews (Clement, 2000) with 20 7th and 8th grade students who were attending an urban public school in a large Midwestern city during the 2014-2015 academic year. The school population, of which the study population mirrored, is approximately 69% African American/Black, 16% Hispanic/Latino, 13% White, and 2% multiracial, with slightly over 85% of students qualifying for free or reduced lunch. The study consisted of three hour-long interviews. The first interview was a selection interview and helped identify the multiplicative concept students were using as well as identified potential productive challenges in the domain of NLMM and NLG. Using the information gathered in the first interview, the second and third interview explored generalizations that students using each of the three different multiplicative concepts made. There were seven tasks in the selection interview protocol, however, tasks were altered based on the interactions with individual students in order to gain authentic insight into students' thinking processes (Clement, 2000). All interviews were video recorded and conducted in the presence of two researchers. Following the interviews, written notes were taken and discussion within the research team served as a form of data triangulation.

Results

We present two data excerpts from the first interview in order to illustrate how we identified what kind of generalizations would be productive for us to work on with MC2 and MC3 students

during the second interview of the study. The first data excerpt is from an MC2 student, Keon, who was presented with the digit problem, which he solved and then created a seven by seven array. He was then asked how many new coordinate points there would be if he was given an eighth digit. The interviewer asked Keon to determine a solution to this problem without initially drawing the coordinate points on the array. This condition was presented by the interviewer in order to test the extent to which Keon could monitor the number of new coordinate points he created as he created them, and to see how he structured his creation of the new coordinate points. Moreover, the goal was to avoid having him put the points on the array and then count the number of points he had put in the array because such a solution had the potential to significantly simplify the problem because it would be less clear in this case if the points retained their status as two-dimensional coordinate points or if they were simply enumerated as units of one.

Excerpt 1: Keon's Solution to Adding a Single Digit

- I:* Would you say what all of them (new coordinate points) are so you can figure out how many there would be?
- K:* Eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven. Yeah. It's eight times seven, right?
-
- I:* Would there be any others?
- K:* Oh, and eight, eight.
- I:* Eight, eight. Yeah that is good. So how many total is that?
- K:* Just the new ones? It would be eight, right?
- I:* Okay. So you said eight, one; eight, two; eight, three...Are there any others you could get?
- K:* [sits in concentration for 5 seconds]: Oh, one, eight.
- I:* Mm-hmm.
- K:* Two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight; and eight, eight.
- I:* Mm-hmm. So how many total new ones would there be?
- K:* It'd be sixteen, right? No. It'd be sixteen, right? Cause it'd be eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven; eight, eight [puts up each finger on his right hand and his thumb, index finger, and middle finger on his left hand as he says the pairs]. One, eight; two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight; eight, eight [puts up his ring finger and pinky on his left hand, and then reuses the fingers on his right hand and the thumb on his left hand as he says each pair]. So sixteen.
- I:* Sixteen. Did you count any of them twice?
- K:* Huh?
- I:* Did you count any of them twice? Check and just see.
- K:* Count any of them twice?
- I:* Mm-hmm. Say them again, and just think about if you counted any of them twice.
- K:* Eight, one; eight, two; eight, three; eight, four; eight, five; eight, six; eight, seven; eight, eight.
- I:* Mm-hmm and what were the other ones you counted?
- K:* One, eight; two, eight; three, eight; four, eight; five, eight; six, eight; seven, eight. Oh, and then eight, eight, I can't say that one no more cause I already said it.
- I:* Yeah that is right.
- K:* So it'd be fifteen.

Keon's initial response in which he said aloud seven new coordinate points (eight, one through eight, seven), and then said, "It's eight times seven, right?" indicated that he anticipated that the problem would involve linear growth. That is, he anticipated that if he added one new digit this

would produce seven new coordinate points because when he had created his seven by seven array each digit produced seven coordinate points. Upon questioning, he realized that eight could also go with itself, and that this meant he produced eight new coordinate points. Moreover, after he produced the coordinate point eight, eight, he had the insight (with the support of further questioning) that eight could also be the second digit in a coordinate point, and was able to state the rest of the coordinate points that he could create.

After he had stated all of the coordinate points, he thought that there would be sixteen coordinate points although was not positive, “It’d be sixteen, right? No.”, and so restated all of them again counting each coordinate point using his fingers as he said them, and concluded there would be a total of sixteen coordinate points. This portion of the data excerpt is of interest for two interrelated reasons: first, Keon was not totally certain about how many new coordinate points he had created and so stated them again in order to count them; and second, he double counted the eight, eight coordinate point, an issue that has been reported frequently in prior research on students’ two-dimensional reasoning (Battista, 2007). He was subsequently able to conclude that he had double counted the eight, eight coordinate point after questioning on the part of the interviewer, and re-stating all of the coordinate points again, to check to see which, if any, of them he had counted twice.

We account for these features of his solution by appealing to the levels of units coordination that are likely involved for a student to immediately conclude that, for example, he had created sixteen new pairs, and that this number of new pairs contained the coordinate point eight, eight twice. That is, our inference is that Keon created the first eight coordinate points as a unit of eight pairs in activity, and that the creation of such a unit structure is equivalent to establishing a three level of unit structure in activity because each of the coordinate points can be considered equivalent to a unit that contains two units (and so a unit of eight pairs is like a unit of eight units of two units). Keon could create a unit of eight pairs in activity but could not operate further with this unit structure; had he been able to operate further with this unit structure our inference is that he would have simply combined the first unit of eight pairs he created with the second unit of eight pairs he created to determine the total number of new coordinate points to be a unit of sixteen pairs without re-creating and counting each of the pairs. Additionally, our inference is that he would have identified, without re-creating all of the pairs, that each unit of eight pairs contained the coordinate point eight, eight.

Nonetheless, this data excerpt suggested to us that for Keon, and students like him, an appropriate and productive challenge would be considering the relationship between a change in the number of digits and the number of coordinate points as a single digit was added. Moreover, we identified that a productive challenge would be structuring the new coordinate points as seven new coordinate points that contained eight as the first digit *only*, seven new coordinate points that contained eight as the second digit *only*, and one new coordinate point that contained eight as both the first and second digit.

The second data excerpt is from an MC3 student, Armando, who was also presented with the digit problem, which he solved and then created a seven by seven array. He was then asked how many times the number of coordinate points he could make if he had twice as many digits. The goal of the interviewer was to have Armando use his array to quantitatively establish the relationship between the number of old and new coordinate points. Once Armando had accomplished this goal, then the interviewer wanted to determine whether Armando could symbolize the situation as $14 \times 14 = (7 \times 2) \times (7 \times 2) = 7^2 \times 2^2$ so that he could also use his symbolic statement to see why there were four times as many coordinate points.

Excerpt 2: Armando’s Solution for Doubling the Number of Digits

I: [A extends the axes of the array, and puts in the digits eight through fourteen on each axes]
How many times more numbers are you going to have if you fill those all in?

- A: Um...can I just write it out?
- I: Well what are you going to do?
- A: Multiply fourteen times fourteen.
- I: Yeah okay so I want you to try to use your picture to figure it out.
- A: Hmm, okay. How can you use the picture?
- I: That is a good question. So like this would be like one time, right? [Points to the seven by seven array that is filled in].
- A: Mm-hmm.
- I: And how many were in here [circles the 7 x 7 part of the array]?
- A: Forty-nine.
- I: Are there going to be any equivalent sections that are going to be this size in your picture?
- A: No ish. Wait yes.
- I: Okay so make another one that is going to be the same size as this.
- A: Hmm, I could do...hmm. I just got this small picture in my head in some way. So since there are seven xs over here [points to the digits eight through fourteen on the x-axis of the array] and going up is seven [points to the digits one through seven on the y-axis] all this area over here is actually also going to be forty-nine[draws the box in the lower right corner].
- I: Okay good.
- A: And the same thing with up here [puts a square in the upper right], and also over here [puts a square in the upper left]. So there is forty-nine four times.
- ...
- I: What were you going to multiply when I said to use the picture?
- A: Oh, I was going to multiply fourteen by fourteen.
- I: Okay so will you write horizontally fourteen times fourteen. And that is equal to? How many times did you have seven in fourteen?
- A: Once, er, what do you mean? Oh! Twice.
- I: Twice here. So can you write fourteen as seven times. Fourteen is seven times what.
- A: Equals seven times two [writes 7×2].
- I: And what about on this side.
- A: Um, same thing [writes $7 \times 2 \times 7 \times 2$, and then $(7 \times 2) \times (7 \times 2)$ so that he has written on his paper that $14 \times 14 = (7 \times 2) \times (7 \times 2)$]
- I: Do you see any number squared in your picture?
- A: Ah, seven squared.
- I: And then do you see some other number squared in your picture?
- A: The other forty nines. Those would also be seven squared [writes $7^2 + 7^2 + 7^2 + 7^2$, and now has written on his paper that $14 \times 14 = (7 \times 2) \times (7 \times 2) = 7^2 + 7^2 + 7^2 + 7^2$].
- ...
- I: And how many seven squareds did you say you have?
- A: Four.
- I: So rather than adding them. What could you do?
- A: Seven squared times four [writes 4×7^2 and sets it equal to what he already has on his paper].
- I: The four is that any number squared?
- A: Two.
- I: Do you see two squared somewhere in your picture, and if so where?
- A: Would it be that [points to the digits one through four on the horizontal axis]?
- I: Okay so say a little bit more.
- A: Like four is two squared. Oh, since, um, eh. Since this right here would be four squared, I mean two squared [circles the four coordinate points in the bottom left of his array].

Our interpretation of this excerpt is that Armando established multiple three level of unit structures in his solution of the problem: he established the fourteen digits on each side of the array as a unit of 2 units of 7 units, and each of the four regions contained in the array as a unit of 7 units of 7 pairs. We make this interpretation because of how he operated to determine that the 14 by 14 array would contain forty-nine coordinate points four times, and the way he subsequently symbolized his reasoning. He first established that there would be a second region that would contain forty-nine coordinate points by identifying that the digits eight through fourteen on the x-axis could be paired with the digits one through seven on the y-axis, which he envisioned could create another region in the array that contained forty nine coordinate points. He then envisioned two other similar 7 by 7 regions. The fact that he could establish these regions without actually having to create any of the coordinate points provided initial indication that he could take these regions as something to operate with, and so was treating the forty-nine coordinate points as a unit of 7 units of 7 pairs.

The assertion that he was operating with three level of unit structures is also supported by how he symbolized the problem: he was able to see 14×14 as a product of 7×2 and 7×2 . In particular, once the interviewer supported him to consider that 14 was equal to 7 times 2, he independently contributed writing 14×14 as equal to the product of 7×2 and 7×2 . This way of symbolizing the problem provides indication that after he established fourteen as containing two sevens he could envision operating further with the two sevens by taking the product of two sevens and two sevens. It is important that he considered this to be a product, and not a sum, because considering it a product was essential for him to see that the product was equivalent to $7^2 + 7^2 + 7^2 + 7^2$: each of seven digits could be paired with seven other digits to produce a region that contained 7^2 coordinate points. Our contention is that to see two sevens and two sevens as a product required maintaining 14 as a unit of 2 units of 7 units because it entailed envisioning operating further with each seven, namely each of seven digits could be paired with another seven digits to make coordinate points without actually having to carry out these operations.

An interesting feature of Armando's solution is that he did not constitute the product as $2^2 \times 7^2$. The interviewer attempted to find out whether he saw the product in this way at first with an indirect question: "I: Do you see any number squared in your picture? A: Ah, seven squared. I: And then do you see some other number squared in your picture? A: The other forty nines. Those would also be seven squared." The interviewer intended the second question to be about whether Armando saw 2^2 in his picture, but Armando answered that what he saw were the other regions in the array that were 7^2 . The interviewer then tried to ask Armando more directly about this issue once Armando had written that the array was equal to 4×7^2 : "I: The four is that any number squared? A: Two." Computationally Armando was able to state that two squared was four, but when asked to identify where this would be in his picture he initially marked the digits from one to four on the x-axis, and then circled the four coordinate points in the lower left corner of his array—indicating that he did not see the four 7^2 sections of the array as 2^2 .

Discussion

The initial interviews in the study have helped us to identify what kind of work is challenging for MC2 and MC3 students related to NLMM and NLG, and thus the kind of problems that are likely to produce interesting generalizations for these students. For MC2 students, the data suggests that problems in which students are asked to consider how the number of coordinate points changes as the number of digits is increased by one are likely to be challenging, but solvable. We think that the potential exists for these students to consider that, for example, $8^2 = 7^2 + 7 + 7 + 1$, and $9^2 = 8^2 + 8 + 8 + 1$, to understand these symbolic statements show that the new number of coordinate points is equal to the old number of coordinate points plus the change in coordinate points where the change in coordinate points is structured as 7, 7, and 1 based on classifying them according to which position the new digit appears in the new coordinate point, and to generalize this understanding.

For MC3 students, the data suggests that problems in which they are asked to consider how multiplicatively changing the number of digits changes the number of coordinate points are likely to be challenging, but solvable. In particular, the challenge appears to lie in seeing that, for example, a 14 by 14 array can be structured as $7^2 \times 2^2$. We conjecture that a problem like the following may support them to establish this way of seeing an array.

Two Color Digit Problem: You have a set of cards where the numerals 1 through 7 are written in blue. You have a second set of cards where the numerals 1 through 7 are written in orange. You select a card, replace it, and select a second card. What different color combinations could you get? What coordinate points could you get in each color combination?

We make this conjecture based on the fact that the problem includes two levels of pairing—pairing colors and pairing digits within a particular color combination. We make this conjecture because pairing digits seemed integral to Armando establishing the 49 coordinate points as 7^2 . We expect that from this type of problem students will be able to generalize that given n colors and x digits in each color that $(nx)^2 = n^2x^2$. We will report on the generalizations that students made in these contexts as part of the presentation.

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